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Analytical and non-analytical corrections to finite-size scaling

P Reinicke

Physikalisches Institut, Universität Bonn, Nussallee 12, D-5300 Bonn 1, Federal Republic of Germany

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Abstract. Using conformal invariance we calculate finite-size corrections to the scaled spectra of one-dimensional quantum chains at the critical point. An explicit application to the Ising and three-state Potts model gives good agreement with analytical and numerical results, respectively.

1. Introduction

In this paper we consider one-dimensional quantum chains, which are related to the transfer matrix of two-dimensional spin systems (Fradkin and Susskind 1978, Kogut 1979). Suppose the Hamiltonian of such a quantum chain with an infinite number of sites N to be conformally invariant at the critical point. If one succeeds in determining the central charge c of this system, then for $c<1$ all possible (not every representation has to be realised) energy eigenvalues are known at the critical point for N infinite (Belavin *et al* 1984, Friedan *et al* 1984). The aim of this paper is to calculate finite-size corrections to such a spectrum (notice that we stay at the critical point).

Let us first review some known results. The spectrum of a quantum chain at the critical point in the finite-size scaling limit is given by certain products of two irreducible representations (IR) Δ and $\bar{\Delta}$ of two commuting Virasoro algebras with the same central charge c (Friedan *et al* 1984). We denote by Δ the highest weight, and by $\Delta+r$, the rth level having degeneracy $d(\Delta, r)$ of one IR of the Virasoro algebra. (The degeneracies $d(\Delta, r)$ can be computed using the character formulae of Rocha-Caridi (1985).) A state will be labelled by $|\Delta+r, \bar{\Delta}+\bar{r}; i\rangle$, $(i=1,2,\ldots,d(\Delta,r)d(\bar{\Delta},\bar{r}))$, so that, at the critical point, we have for the scaled energy gaps as the number of sites N goes to infinity

$$\mathscr{F}(\Delta+r, \bar{\Delta}+\bar{r}; i) \equiv (N/2\pi)(E(\Delta+r, \bar{\Delta}+\bar{r}; i)-E(0,0)) \xrightarrow[N\to\infty]{} \Delta+r+\bar{\Delta}+\bar{r} \tag{1.1}$$

where we omit the index i if the state is non-degenerate and E denotes the energy. In the same limit we have for the scaled ground-state energy (Affleck 1986, Blöte *et al* 1986)

$$\mathscr{E}_0 \equiv (N/2\pi)(E(0,0)-a_0N) \xrightarrow[N\to\infty]{} -c/12 \tag{1.2}$$

where a_0 is a non-universal constant.

For finite N one expects instead of (1.1) and (1.2)

$$\mathscr{F}(\Delta+r, \bar{\Delta}+\bar{r}, i) = \Delta+r+\bar{\Delta}+\bar{r}+c_1(\Delta, \bar{\Delta}, r, \bar{r}; i)N^{-\alpha_1}+c_2(\Delta, \bar{\Delta}, r, \bar{r}; i)N^{-\alpha_2}+\ldots \tag{1.3}$$

$$\mathscr{E}_0 = -c/12+\tilde{c}_1N^{-\alpha_1}+\tilde{c}_2N^{-\alpha_2}+\ldots \tag{1.4}$$

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where $0<\alpha_1<\alpha_2$ and the functions $c_1, c_2, \ldots$, are unknown. Our aim is to clarify the values of α and c. If α is an integer we have analytic corrections, otherwise not.

The paper is organised as follows. In § 2 we calculate some finite-size corrections in (1.3) and (1.4) for the case where the state $|\Delta+r, \bar{\Delta}+\bar{r}; i\rangle$ is non-degenerate. In §§ 3 and 4 we apply our results to the Ising model and the three-state Potts model and compare them with the exact solution and numerical results, respectively. A summary of our results is presented in § 5.

2. Finite-size corrections due to operators belonging to the conformal block of the unit operator (analytic corrections)

Consider a strip of width N with periodic or twisted boundary conditions. We want to calculate corrections to the spectrum of the conformal theory for N large. The Hamiltonian will differ from the fixed point Hamiltonian by terms involving irrelevant operators, so that for N large enough one has (Cardy 1986a)

$$H = H^{c} + \sum_j \beta_j \int_{-N/2}^{N/2} \mathrm{d}v\, \phi_j(0, v) \tag{2.1}$$

where β_j are unknown constants and ϕ_j are local fields of the conformal theory. A field ϕ on the strip depends on the variables $w=\tau+\mathrm{i}v$ and $\bar{w}=\tau-\mathrm{i}v$ ($-\infty<\tau<\infty, -N/2\leqslant v\leqslant N/2$). Sometimes we write $\phi(\tau, v)$ instead of $\phi(w, \bar{w})$. The choice $\tau=0$ in (2.1) is arbitrary due to translation invariance in the τ direction. Let $(\Delta_j+r_j, \bar{\Delta}_j+\bar{r}_j)$ be the scaling dimension of ϕ_j and let

$$x_j = \Delta_j + r_j + \bar{\Delta}_j + \bar{r}_j. \tag{2.2}$$

One can show that the field ϕ_j will give, in k-order perturbation, corrections to the gap (1.3) proportional to $N^{-k(x_j-2)}$. If $\phi(w)$ is a primary field then $(L_{-k}\phi)(w)$ denotes a field belonging to its tower, especially $L_{-k}(w)$ belongs to the tower of the identity. In this section we investigate corrections up to order N^{-2}, coming from the tower of the identity ($\Delta_j=\bar{\Delta}_j=0$ in (2.2)). We start with a lemma.

Lemma. The operators $\int_{-N/2}^{N/2}\mathrm{d}v\, L_{-k}(w)$, $k\geqslant 3$, give no corrections in any order of perturbation theory to the fixed point Hamiltonian.

Proof. It is sufficient to show that all matrix elements vanish. Consider

$$\begin{aligned}
&\langle\Delta+r, \bar{\Delta}+\bar{r}; i| \int_{-N/2}^{N/2} \mathrm{d}v\, L_{-k}(w) |\Delta'+r'; \bar{\Delta}'+\bar{r}'; j\rangle \\
&\quad = \delta_{\Delta,\Delta'}\delta_{\bar{\Delta},\bar{\Delta}'}\delta_{\bar{r},\bar{r}'} \int_{-N/2}^{N/2} \mathrm{d}v \exp[(2\pi/N)(r'-r)(\tau+\mathrm{i}v)] \\
&\qquad \times \langle\Delta+r, \bar{\Delta}+\bar{r}; i|L_{-k}(0)|\Delta+r', \bar{\Delta}+\bar{r}; j\rangle \\
&\quad = N\delta_{\Delta,\Delta'}\delta_{\bar{\Delta},\bar{\Delta}'}\delta_{r,r'}\delta_{\bar{r},\bar{r}'}\langle\Delta+r, \bar{\Delta}+\bar{r}; i|L_{-k}(0)|\Delta+r, \bar{\Delta}+\bar{r}; j\rangle.
\end{aligned} \tag{2.3}$$

Now from

$$L_{-k}(w) \equiv \frac{1}{2\pi\mathrm{i}} \oint_{c_w} \frac{\mathrm{d}\eta\, T(\eta)}{(\eta-w)^{k-1}} = [(k-2)!]^{-1}\partial_w^{k-2}T(w)$$

(the counter c_w surrounds the point w) one has for $k \geqslant 3$

$$\langle \Delta + r, \bar{\Delta} + \bar{r};\, i | L_{-k}(w) | \Delta + r, \bar{\Delta} + \bar{r};\, j \rangle$$
$$= [(k-2)!]^{-1} \partial_w^{k-2} \langle \Delta + r, \bar{\Delta} + \bar{r};\, i | T(w) | \Delta + r, \bar{\Delta} + \bar{r};\, j \rangle = 0 \tag{2.4}$$

since the last matrix element is w-independent. This completes the proof.

Having in mind that we are interested only in corrections up to N^{-2} to the gaps (1.3), we are left with the two possibilities

$$\phi_1(w, \bar{w}) = L_{-2}(w) \bar{L}_{-2}(\bar{w}) \tag{2.5}$$

and

$$\phi_2(w, \bar{w}) = L_{-2}^2(w) + \bar{L}_{-2}^2(\bar{w}) \tag{2.6}$$

if we suppose that ϕ_j belongs to the tower of the identity. (The combination $L_{-2}^2(w) - \bar{L}_{-2}^2(\bar{w})$ is not possible, since it does not respect the symmetry $E(p) = E(-p)$.)

In order to calculate the matrix elements $\langle \Delta + r, \bar{\Delta} + \bar{r} | \phi_k(0, 0) | \Delta + r, \bar{\Delta} + \bar{r} \rangle$—we suppose that the state $(\Delta + r, \bar{\Delta} + \bar{r})$ is non-degenerate—consider the spectral decomposition of the three-point function

$$\langle \phi_{\Delta,\bar{\Delta}}(\tau_1, v_1) \phi_k(\tau_2, v_2) \phi_{\Delta,\bar{\Delta}}(\tau_3, v_3) \rangle$$
$$= \sum_{\substack{r_1, r_2 \\ \bar{r}_1, \bar{r}_2}} \sum_{i,j} \langle 0, 0 | \phi_{\Delta,\bar{\Delta}}(0, 0) | \Delta + r_1, \bar{\Delta} + \bar{r}_1;\, i \rangle$$
$$\times \langle \Delta + r_1, \bar{\Delta} + \bar{r}_1;\, i | \phi_k(0, 0) | \Delta + r_2, \bar{\Delta} + \bar{r}_2;\, j \rangle$$
$$\times \langle \Delta + r_2, \bar{\Delta} + \bar{r}_2;\, j | \phi_{\Delta,\bar{\Delta}}(0, 0) | 0, 0 \rangle \xi_1^{\Delta + r_1} \xi_2^{\Delta + r_2} \bar{\xi}_1^{\bar{\Delta} + \bar{r}_1} \bar{\xi}_2^{\bar{\Delta} + \bar{r}_2}$$
$$= \sum_{\substack{r_1, r_2 \\ \bar{r}_1, \bar{r}_2}} a_{r_1, r_2; \bar{r}_1, \bar{r}_2} \xi_1^{\Delta + r_1} \xi_2^{\Delta + r_2} \bar{\xi}_1^{\bar{\Delta} + \bar{r}_1} \bar{\xi}_2^{\bar{\Delta} + \bar{r}_2} \tag{2.7}$$

and that of the two-point function

$$\langle \phi_{\Delta,\bar{\Delta}}(\tau_1, v_1) \phi_{\Delta,\bar{\Delta}}(\tau_3, v_3) \rangle$$
$$= \sum_{r, \bar{r}, i} \langle 0, 0 | \phi_{\Delta,\bar{\Delta}}(0, 0) | \Delta + r, \bar{\Delta} + \bar{r};\, i \rangle \langle \Delta + r, \bar{\Delta} + \bar{r};\, i | \phi_{\Delta,\bar{\Delta}}(0, 0) | 0, 0 \rangle$$
$$\times (\xi_1 \xi_2)^{\Delta + r} (\bar{\xi}_1 \bar{\xi}_2)^{\bar{\Delta} + \bar{r}}$$
$$= \sum_{r, \bar{r}} b_{r, \bar{r}} (\xi_1 \xi_2)^{\Delta + r} (\bar{\xi}_1, \bar{\xi}_2)^{\bar{\Delta} + \bar{r}} \tag{2.8}$$

where

$$\xi_j = \exp\{(2\pi/N)[(\tau_j - \tau_{j+1}) + \mathrm{i}(v_j - v_{j+1})]\}. \tag{2.9}$$

From comparison one obtains for $(\Delta + r, \bar{\Delta} + \bar{r})$ non-degenerate

$$\langle \Delta + r, \bar{\Delta} + \bar{r} | \phi_k(0, 0) | \Delta + r, \bar{\Delta} + \bar{r} \rangle = a_{r, r; \bar{r}, \bar{r}} / b_{r, \bar{r}}. \tag{2.10}$$

Formulae (2.7) and (2.8) are valied for Δ and $\bar{\Delta}$ different from zero. The case $\Delta = \bar{\Delta} = 0$ is obtained by replacing $\phi_{\Delta,\bar{\Delta}}(w, \bar{w})$ by $T(w) \bar{T}(\bar{w})$ in formulae (2.7) and (2.8). If, let us say, Δ is zero and $\bar{\Delta}$ is not then one has to treat the w and $\bar{w}$ dependence separately.

In order to obtain the corrections to the state $(\Delta+r, \bar{\Delta}+\bar{r})$—which is supposed to be non-degenerate—due to the operators (2.5) and (2.6), we need the matrix elements

$$\begin{aligned}\langle\Delta+r, \bar{\Delta}+r|L_{-2}\bar{L}_{-2}(0,0)|\Delta+r, \bar{\Delta}+\bar{r}\rangle \\ =\langle\Delta+r|L_{-2}(0,0)|\Delta+r\rangle\langle\bar{\Delta}+\bar{r}|\bar{L}_{-2}(0,0)|\bar{\Delta}+\bar{r}\rangle\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}\langle\Delta+r, \bar{\Delta}+\bar{r}|L^2_{-2}(0,0)+\bar{L}^2_{-2}(0,0)|\Delta+r, \bar{\Delta}+\bar{r}\rangle \\ =\langle\Delta+r|L^2_{-2}(0,0)|\Delta+r\rangle+\langle\bar{\Delta}+\bar{r}|\bar{L}^2_{-2}(0,0)|\bar{\Delta}+\bar{r}\rangle\end{aligned} \tag{2.12}$$

respectively.

For $\Delta=0$ one obtains from

$$\begin{aligned}\langle L_{-2}(w_1)L_{-2}(w_2)L_{-2}(w_3)\rangle \\ &=\left(\frac{2\pi}{N}\right)^6\left(\frac{1}{2\pi \mathrm{i}}\right)^3\oint_{C_{z_1}}\mathrm{d}\eta_1\oint_{C_{z_2}}\mathrm{d}\eta_2\oint_{C_{z_3}}\mathrm{d}\eta_3 \\ &\quad\times\left\langle\prod_{i=1}^{3}\left(\eta_i T(\eta_i)-\frac{c}{24\eta_i}\right)\right\rangle\left(\prod_{j=1}^{3}\ln\frac{\eta_j}{z_j}\right)^{-1} \\ &=\left(\frac{2\pi}{N}\right)^6\left[\frac{c(\xi_1\xi_2)^2}{(1-\xi_1)^2(1-\xi_2)^2(1-\xi_1\xi_2)^2}\right. \\ &\quad\left.-\frac{c^2}{48}\left(\frac{\xi_1^2}{(1-\xi_1)^4}+\frac{\xi_2^2}{(1-\xi_2)^4}+\frac{(\xi_1\xi_2)^2}{(1-\xi_1\xi_2)^4}\right)-\left(\frac{c}{24}\right)^3\right]\end{aligned} \tag{2.13}$$

and

$$\langle L_{-2}(w_1)L_{-2}(w_3)\rangle=\left(\frac{2\pi}{N}\right)^4\left(\frac{c}{2}\frac{(\xi_1\xi_2)^2}{(1-\xi_1\xi_2)^4}+\left(\frac{c}{24}\right)^2\right) \tag{2.14}$$

the element

$$\langle r|L_{-2}(0,0)|r\rangle=(2\pi/N)^2(r-c/24) \qquad (r\neq 1). \tag{2.15}$$

Following the lines of (2.7)-(2.9) one obtains for Δ different from zero

$$\langle\Delta+r|L_{-2}(0,0)|\Delta+r\rangle=(2\pi/N)^2/(\Delta+r-c/24). \tag{2.16}$$

After some tedious calculations one obtains in a similar way

$$\langle\Delta+r|L^2_{-2}(0,0)|\Delta+r\rangle=\left(\frac{2\pi}{N}\right)^4\left[\left(\frac{c}{24}\right)^2+\frac{11c}{1440}+A(\Delta,r)\right] \tag{2.17}$$

where

$$\begin{aligned}A(0,r)&=\left(\frac{11}{30}+\frac{c}{12}\right)r(2r^2-3) \qquad (r\neq 1) \\ A(\Delta,r)&=(\Delta+r)\left(\left(\Delta-\frac{1}{6}-\frac{c}{12}\right)+\frac{r(2\Delta+r)(5\Delta+1)}{(\Delta+1)(2\Delta+1)}\right) \qquad \Delta\neq 0.\end{aligned} \tag{2.18}$$

Summing up our results we have for ϕ_j belonging to the tower of the identity in (2.1)

$$H=H^c+\beta_1\int_{-N/2}^{N/2}\mathrm{d}v(L_{-2}\bar{L}_{-2})(0,v)+\beta_2\int_{-N/2}^{N/2}\mathrm{d}v(L^2_{-2}(0,v)+\bar{L}^2_{-2}(0,v))+\dots \tag{2.19}$$

the following corrections to the energy gap and the scaled ground-state energy:

$$\mathcal{F}(\Delta+r, \bar{\Delta}+\bar{r}) = \Delta+r+\bar{\Delta}+\bar{r}+\frac{8\pi^3}{N^2}\left\{\beta_1\left[\left(\Delta+r-\frac{c}{24}\right)\left(\bar{\Delta}+\bar{r}-\frac{c}{24}\right)-\left(\frac{c}{24}\right)^2\right]\right.$$
$$\left.+\beta_2[A(\Delta, r)+A(\bar{\Delta}, \bar{r})]\right\}+\mathrm{O}(N^{-4}) \quad (2.20)$$

$$\mathscr{E}_0 = -\frac{c}{12}+\frac{\pi^3}{N^2}\left[\beta_1\frac{c^2}{72}+\beta_2\left(\frac{c^2}{36}+\frac{11c}{90}\right)\right]+\mathrm{O}(N^{-4}). \quad (2.21)$$

Notice that one also has to consider corrections coming from local fields ϕ_j that do not belong to the tower of the identity. In order to do this one has to specify the model first.

3. Applications to the Ising model

On a chain with N sites the Hamiltonian of the Ising model is (Katsura 1962)

$$H = -\frac{\lambda}{2\gamma}\sum_{n=1}^{N}\sigma^z(n)-\frac{1}{4\gamma}\sum_{n=1}^{N}[(1+\gamma)\sigma^x(n+1)\sigma^x(n)+(1-\gamma)\sigma^y(n+1)\sigma^y(n)] \quad (3.1)$$

where σ^x, σ^y and σ^z are the Pauli matrices. The normalisation of H, which is in principle arbitrary, is fixed by the requirements of conformal theory (von Gehlen *et al* 1986). The phase diagram is well known (Barouch and McCoy 1971). For all $\gamma(0<\gamma\leqslant 1)$, there is a critical point at $\lambda_c=1$, which falls into the 2D Ising universality class (for $\gamma=1$ one has the Ising model). At the critical point ($\lambda_c=1$) and infinite size the spectra of the Hamiltonian (3.1) with periodic and antiperiodic boundary conditions are built by the irreducible representations $(\Delta, \bar{\Delta})$ of two commuting Virasoro algebras L_n and $\bar{L}_n$ with central charge $c=\frac{1}{2}$ (Cardy 1986b, Henkel 1987). The possible values of $(\Delta, \bar{\Delta})$ are $(0,0)$, $(\frac{1}{2},\frac{1}{2})$, $(\frac{1}{16},\frac{1}{16})$ for periodic and $(0,\frac{1}{2})$, $(\frac{1}{2},0)$, $(\frac{1}{16},\frac{1}{16})$ for antiperiodic boundary conditions.

By expanding the exact solution of (3.1), Henkel (1987) has obtained several corrections to the conformal spectra ($\lambda=1$, $N=\infty$). Let us list his results for the non-degenerate states of the system at the critical point ($\lambda_c=1$)

$$\mathcal{F}(\Delta+r, \bar{\Delta}+\bar{r}) = \Delta+r+\bar{\Delta}+\bar{r}+(\pi^2/2N^2)(1/\gamma^2-\tfrac{4}{3})(a(\Delta, r)+a(\bar{\Delta}, \bar{r}))+\mathrm{O}(N^{-4}) \quad (3.2)$$

$$\mathscr{E}_0 = -\frac{1}{24}+\frac{7\pi^2}{1920N^2}\left(\frac{1}{\gamma^2}-\frac{4}{3}\right)+\mathrm{O}(N^{-4}) \quad (3.3)$$

where

$$\begin{aligned} &a(\tfrac{1}{2}, r) = (r+\tfrac{1}{2})^3 && 0\leqslant r\leqslant 3\\ &a(0, r) = (r-\tfrac{1}{2})^3+\tfrac{1}{8} && r=0,2,3\\ &a(\tfrac{1}{16}, r) = r^3-\tfrac{1}{128} && r=0,1,2. \end{aligned} \quad (3.4)$$

A comparison with (2.20) and (2.21) shows complete agreement for

$$\beta_1 = 0 \qquad \text{and} \qquad \beta_2 = \frac{3}{56\pi}\left(\frac{1}{\gamma^2}-\frac{4}{3}\right). \quad (3.5)$$

Until now we have concentrated our attention only on operators ϕ_j in (2.1) which belong to the tower of the identity. If one allows other operators, one can only take fields belonging to the tower of the energy density ε—that has $\Delta=\bar{\Delta}=\frac{1}{2}$—since other fields do not have charge zero but the Hamiltonian (3.1) does. Therefore one obtains only analytic corrections for the Ising model. Within the tower of the energy density one finds essentially only one operator that gives N^{-2} corrections to the scaled spectra, namely

$$\begin{aligned}\beta_0\int_{-N/2}^{N/2}\mathrm{d}v\,\phi_0(0,v)&=\beta_0\int_{-N/2}^{N/2}\mathrm{d}v(L_{-1}\bar{L}_{-1}\varepsilon)(0,v)\\ &\equiv\beta_0\int_{-N/2}^{N/2}\mathrm{d}v(\partial_w\partial_{\bar{w}}\varepsilon)(0,v)=\beta_0\int_{-N/2}^{N/2}\mathrm{d}v\left(\frac{\partial^2\varepsilon}{\partial\tau^2}\right)(0,v)\\ \int_{-N/2}^{N/2}\mathrm{d}v(L_{-1}^2\varepsilon)(0,v)&=\int_{-N/2}^{N/2}\mathrm{d}v(\bar{L}_{-1}L_{-1}\varepsilon)(0,v)\qquad L_{-2}\varepsilon\equiv\tfrac{3}{4}L_{-1}^2\varepsilon.\end{aligned}\tag{3.6}$$

It is easy to see that in the first-order perturbation theory ϕ_0 gives no contribution, since it is a derivative. We have calculated the second-order corrections to all non-degenerate states and obtained full agreement with (3.2) for

$$\beta_0^2=\frac{1}{16\pi^2}\left(\frac{1}{\gamma^2}-\frac{4}{3}\right).\tag{3.7}$$

Note that β_0 has to be real (hermiticity of the Hamiltonian), which is not the case for $\frac{1}{2}\sqrt{3}\leqslant\gamma\leqslant 1$. Calculating the contribution to the ground-state energy, one encounters a divergent integral, which can be regularised. One can show that the final result is regularisation independent.

Let $V=\beta_0\int_{-N/2}^{N/2}\mathrm{d}v(L_{-1}\bar{L}_{-1}\varepsilon)(0,v)$, then

$$\begin{aligned}\mathscr{E}_0+\frac{c}{12}&=-\frac{N}{2\pi}\sum_{i\neq 0}\frac{\langle 0|V|i\rangle\langle i|V|0\rangle}{E_i^{(0)}-E_0^{(0)}}=-\frac{N}{2\pi}\int_0^\infty\mathrm{d}\tau\sum_i\exp[-(E_i^{(0)}-E_0^{(0)})\tau]\langle 0|V|i\rangle\langle i|V|0\rangle\\ &=-\frac{N}{2\pi}\beta_0^2\int_0^\infty\mathrm{d}\tau\int_{-N/2}^{N/2}\mathrm{d}v_1\int_{-N/2}^{N/2}\mathrm{d}v_2\\ &\quad\times\langle 0|(L_{-1}\bar{L}_{-1}\varepsilon)(0,v_1)(L_{-1}\bar{L}_{-1}\varepsilon)(\tau,v_2|0\rangle\\ &=-\left(\frac{2\pi}{N}\right)^2\beta_0^2\int_0^\infty\mathrm{d}\tau\int_{-\pi}^{\pi}\mathrm{d}v_1\int_{-\pi}^{\pi}\mathrm{d}v_2\left|\xi\partial_\xi\xi\partial_\xi\frac{\xi^{1/2}}{1-\xi}\right|^2\Bigg|_{\xi=\exp[-\tau+\mathrm{i}(v_1-v_2)]}\\ &=-\left(\frac{\pi}{N}\right)^2(4\pi\beta_0)^2\int_0^\infty\mathrm{d}\tilde{\tau}\sum_{\nu=0}^{\infty}(\nu+\tfrac{1}{2})^4\exp[-\tilde{\tau}(\nu+\tfrac{1}{2})]\\ &=\frac{\pi^2}{N^2}(4\pi\beta_0)^2\frac{7}{16}\zeta(-3)=\frac{7\pi^2}{1920N^2}(4\pi\beta_0)^2.\end{aligned}\tag{3.8}$$

One can also introduce a cutoff b ($\int_0^\infty\mathrm{d}\tau\ldots\to\int_b^\infty\mathrm{d}\tau\ldots$), which has the meaning of a lattice constant and obtain

$$\mathscr{E}_0+\frac{c}{12}=\frac{\pi^2}{N^2}(4\pi\beta_0)^2\left[-\frac{3}{16}\left(\frac{N}{2\pi b}\right)^4+\frac{7}{1920}+\mathrm{O}(b/N)\right].$$

The finite part is of course the same as in (3.8). Looking at the N dependence of the divergent part, one sees that it can be absorbed by the regular part of the free energy, i.e. the constant a_0 in (1.2).

Inserting (3.7) into (3.8) one sees that the operator ϕ_0 also gives the right contribution to the ground-state energy. In conclusion we have found that the Hamiltonian operator

$$H = H^c + \beta \int_{-N/2}^{N/2} dv (L_{-1}\bar{L}_{-1}\varepsilon)(0, v)$$
$$+\frac{3}{56\pi}\left[\frac{1}{\gamma^2}-\frac{4}{3}-(4\pi\beta)^2\right]\int_{-N/2}^{N/2} dv(L_{-2}^2(0, v)+\bar{L}_{-2}^2(0, v)) \tag{3.9}$$

where β is an arbitrary real number and as usual

$$H^c = \frac{1}{2\pi}\int_{-N/2}^{N/2} dv(L_{-2}(0, v)+\bar{L}_{-2}(0, v)) + \text{regular terms} \tag{3.10}$$

has the same scaled spectrum as the Hamiltonian operator (3.1) at the critical point up to $O(N^{-4})$. Unfortunately we did not succeed in finding a criterion to fix the parameter β.

4. Applications to the three-state Potts model

Let us remind the reader that the spectra of the three-state Potts model with periodic and twisted boundary conditions are built by the IR of two commuting Virasoro algebras with central charge $c = \frac{4}{5}$ (von Gehlen and Rittenberg 1986, Cardy 1986b).

Since no analytic results are known, we will compare the predictions of (2.20) and (2.21) to the numerical data of von Gehlen *et al* (1987). They found for the energy gaps at the critical temperature the following N dependence:

$$\mathscr{F}(\Delta+r, \bar{\Delta}+\bar{r}) = \Delta+r+\bar{\Delta}+\bar{r}+c_1(\Delta, \bar{\Delta}, r, \bar{r})N^{-0.8}+c_2(\Delta, \bar{\Delta}, r, \bar{r})N^{-2}+\dots \tag{4.1}$$

Furthermore, they were able to explain the $N^{-0.8}$ corrections, namely setting

$$H = H^c + \gamma\int_{-N/2}^{N/2} dv\, \phi_{7/5,7/5}(0, v)+\dots \tag{4.2}$$

where $\gamma = 0.009\,237\,(7)$. Notice that there are non-analytical and analytical corrections in (4.1.).

In table 1 we show their values of $c_2(\Delta, \bar{\Delta}, r, \bar{r})$. For the scaled ground-state energy they obtained

$$\mathscr{E}_0 = -\tfrac{1}{15} - 0.028\,0\,(5)/N^2+\dots \tag{4.3}$$

Table 1. Numerical values of $c_2(\Delta, \bar{\Delta}, r, \bar{r})$ (see von Gehlen *et al* 1987) defined by (4.1) for several levels.

$(\Delta+r, \bar{\Delta}+\bar{r})$	$c_2(\Delta, \bar{\Delta}, r, \bar{r})$
$(0+2, 0)$	−5.810 (1)
$(\frac{1}{15}, \frac{1}{15})$	+0.032 38 (2)
$(\frac{1}{15}+1, \frac{1}{15})$	−1.6 (1)
$(\frac{2}{5}, \frac{2}{5})$	−0.328 (2)
$(\frac{2}{5}, \frac{1}{15})$	−1.5 (3)

Comparing these values to (2.20) and (2.21) we obtain after a weighted fit $\beta_1 = -0.002\,(2)$ and $\beta_2 = -0.005\,6\,(3)$. Since these values are consistent with $\beta_1 = 0$ we set β_1 to zero and obtain

$$\beta_2 = -0.005\,6\,(6). \tag{4.4}$$

If we do not restrict ϕ_j in (2.1) to the tower of the identity, then ϕ_j will have scaling dimensions $(\Delta_j + r_j, \bar{\Delta}_j + \bar{r}_j)$. Since the Hamiltonian operator of the three-state Potts model on a finite quantum chain has charge zero and is neutral under charge conjugation the only possible values of $(\Delta_j \bar{\Delta}_j)$ are $\Delta_j = \bar{\Delta}_j = 0, \frac{2}{5}, \frac{7}{5}$ or 3. (Rittenberg 1986). It follows that N^{-2} corrections to the scaled spectra can only be obtained from the tower of the identity (i.e. $\Delta_j = \bar{\Delta}_j = 0$).

One important problem is still left. From (4.2) we expect, in second order, $N^{-1.6}$ corrections to the gaps, which were not found numerically. We have calculated these corrections for the gap $\mathscr{F}(2,0)$ and the scaled ground-state energy $\mathscr{E}_0$, and obtain after a regularisation

$$\begin{aligned} \mathscr{F}(2,0) &= 2 - 0.010\,93\,(2)/N^{1.6} + \ldots \\ \mathscr{E}_0 &= -\tfrac{1}{15} - 0.001\,721\,(3)/N^{1.6} + \ldots\ldots \end{aligned} \tag{4.5}$$

(The numerical uncertainty is due to the uncertainty of γ in (4.2).) A comparison of these numbers with the coefficient of N^{-2} in (4.1) and (4.3), respectively, explains why von Gehlen *et al* (1987) did not find the N dependence (in calculations with $N \leqslant 12$) shown in (4.5).

Finally we show the explicit calculation of the corrections to the ground state in (4.5). Consider the more general case

$$H = H^{c} + \gamma \int_{-N/2}^{N/2} \mathrm{d}v\, \phi_{\Delta,\bar{\Delta}}(0, v) + \ldots \tag{4.6}$$

where $\Delta = \bar{\Delta} := \frac{1}{2}x$. Then in analogy to (3.8) one obtains in second-order perturbation theory

$$\begin{aligned} \mathscr{E}_0 + \frac{c}{12} &= -\frac{N}{2\pi}\gamma^2 \int_0^{x} \mathrm{d}\tau \int_{-N/2}^{N/2} \mathrm{d}v_1 \int_{-N/2}^{N/2} \mathrm{d}v_2 \\ &\quad \times \left(\frac{2\pi}{N}\right)^{2x} \left|\frac{\xi^{\Delta}}{(1-\xi)^{2\Delta}}\right|^2 \Bigg|_{\xi = \exp\{-(2\pi/N)[\tau + \mathrm{i}(v_2 - v_1)]\}} \\ &= -\left[2\pi\gamma\left(\frac{2\pi}{N}\right)^{x-2}\right]^2 \frac{1}{2}\int_0^1 \mathrm{d}z\, z^{x/2-1} F(x, x; 1; z). \end{aligned} \tag{4.7}$$

The last integral is only convergent for $0 < \operatorname{Re} x < 1$ and we are interested in $x > 2$ (F is the hypergeometric function). For $0 < \operatorname{Re} x < 1$ we have after $2k$ partial integrations

$$\begin{aligned} \int_0^1 \mathrm{d}z\, z^{x/2-1} F(x, x; 1; z) &= -\sum_{\nu=1}^{k} \frac{\Gamma^2(1-x)\Gamma^2(\nu - \frac{1}{2}x)\Gamma(2\nu - 2x)}{\Gamma^2(1-\frac{1}{2}x)\Gamma^3(1+\nu-x)\Gamma(\nu - x)}(3\nu - 2x) \\ &\quad + \frac{\Gamma^2(k+1-\frac{1}{2}x)\Gamma^2(1-x)}{\Gamma^2(1-\frac{1}{2}x)\Gamma^2(k+1-x)} \int_0^1 \mathrm{d}z\, z^{x/2-1} F(x-k, x-k; 1; z). \end{aligned} \tag{4.8}$$

The right-hand side of this equation can be extended to all x with $0 < \operatorname{Re} x < k+1$ and $x \notin N$ (for x half-integer one can show that the sum in (4.8) is well defined). This method, which is obviously regularisation independent, has been used in the calculation of (4.5).

Summing up, we have for the three-state Potts model

$$H = H^c + \gamma \int_{-N/2}^{N/2} dv\, \phi_{7/5,7/5}(0, v) + \beta_2 \int_{-N/2}^{N/2} dv \left(L_{-2}^2(0, v) + \bar{L}_{-2}^2(0, v) \right) + \dots \quad (4.9)$$

where $\gamma = 0.009\,237\,(7)$ and $\beta_2 = -0.005\,6\,(6)$.

5. Conclusions

Using conformal invariance at the critical point we have calculated finite-size corrections to the spectrum of a one-dimensional quantum chain in the finite-size scaling limit. Considering in (2.1) only local fields ϕ_j, that belong to the tower of the identity, we have obtained finite-size corrections for the scaled spectrum of non-degenerate levels up to $O(N^{-4})$ (of course, one also expects, in general, corrections due to fields ϕ_j that do not belong to the tower of the identity). With this frame one obtains only corrections proportional to N^{-2} due to the fields $\phi_1 = L_{-2}\bar{L}_{-2}$ and $\phi_2 = L_{-2}^2 + \bar{L}_{-2}^2$, which are given in (2.18)–(2.21). For the Ising and the three-state Potts models we have compared our results with the exact solution and numerical data, respectively, and conclude that the field ϕ_1 does not appear in (2.1) in both cases.

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References

Affleck I 1986 *Phys. Rev. Lett.* **56** 746
Barouch E and McCoy B M 1971 *Phys. Rev.* A **3** 786
Belavin A A, Polyakov A M and Zamolodchikov A B 1984 *Nucl. Phys.* B **241** 333
Blöte H W, Cardy J L and Nightingale M P 1986 *Phys. Rev. Lett.* **56** 742
Cardy J L 1986a *Phase Transitions and Critical Phenomena* vol 11, ed C Domb and J L Lebowitz (New York: Academic)
—— 1986b *Nucl. Phys.* B **275** 200
Franklin E and Susskind L 1978 *Phys. Rev.* D **17** 2637
Friedan D, Quiu Z and Shenker S 1984 *Phys. Rev. Lett.* **52** 1575
von Gehlen G and Rittenberg V 1986 *J. Phys. A: Math. Gen.* **19** L625
von Gehlen G, Rittenberg V and Ruegg H 1986 *J. Phys. A: Math. Gen.* **19** 107
von Gehlen G, Rittenberg V and Vescan T 1987 *J. Phys. A: Math. Gen.* **20** 2577
Henkel M 1987 *J. Phys. A: Math. Gen.* **20** 995
Katsura S 1962 *Phys. Rev.* **127** 1508
Kogut J 1979 *Rev. Mod. Phys.* **51** 659
Rittenberg V 1986 Private communication
Rocha-Caridi A 1985 *Vertex Operators in Mathematics and Physics* ed J Lepovsky (Berlin: Springer) p 451